The harmonic oscillator and Coulomb potentials - two exceptions from the point of view of a function theory

1996 J. Phys. A: Math. Gen. 29 3203

(http://iopscience.iop.org/0305-4470/29/12/024)

J. Phys. A: Math. Gen. **29** (1996) 3203–3207. Printed in the UK

The harmonic oscillator and Coulomb potentials—two exceptions from the point of view of a function theory

I A Ivanov

Institute of Spectroscopy, Academy of Sciences of Russia, 142092, Troitsk, Moscow Region, Russia

Received 30 January 1996

Abstract. We show that the Coulomb potential, one- and three-dimensional harmonic oscillator potentials are the only potentials belonging to a certain class which have the following property. The Schrödinger equation has infinitely many eigenvalues belonging to the discrete spectrum with the eigenfunctions having only a finite number of zeros in a complex plane. We show that it is due to this fact that these two potentials are the only potentials, belonging to the class defined in the paper, for which the semiclassical quantization gives an energy spectrum coinciding with the results of an exact quantum mechanical treatment.

It is well known that the potential of the harmonic oscillator and the Coulomb potential exhibit a number of peculiar properties both in the classical and quantum domains. In classical mechanics these two potentials are the only potentials permitting the closed trajectories of a particle revolving around the centre of attraction (Landau and Lifshitz 1971). In quantum mechanics these two potentials are also singled out. One can mention the well known fact that the semiclassical approximation applied to the harmonic oscillator and the Coulomb problem gives an exact result in both cases. In the present paper we show that the harmonic oscillator and the Coulomb potentials are the only potentials belonging to a certain class of potentials (defined below) for which the semiclassical approximation gives an exact energy spectrum. This property is connected with certain function theoretical properties of the bound-state wavefunctions. We shall show that the harmonic oscillator and the Coulomb potentials are the only potentials belonging to a certain class, for which the bound-state wavefunctions have only a finite number of zeros in the complex plane.

We consider the one-dimensional Schrödinger equation

$$-\frac{1}{2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+U(x)y=Ey \tag{1}$$

where x can vary in the intervals $[0,+\infty)$ or $(-\infty,\infty)$. The potential $U(x)$ in equation (1) has the following form,

$$U(x)=\frac{l(l+1)}{2x^2}+\frac{f(x)}{x} \tag{2}$$

where l is an integer non-negative parameter, and $f(x)$ is an entire function of a variable x. If eigenvalue problem (1) is posed on the interval $[0,\infty)$, we search for the solution $y(x)$ behaving as x^{l+1} near the origin. If in (1), $x\in(-\infty,+\infty)$, then in (2) the parameter $l=0$ and we search for the solution $y(x)$ regular at the origin. Since the potential $U(x)$ in

0305-4470/96/123203+05$19.50

(1) is an analytic function of x we can consider the solution $y(x)$ as an analytic function in a whole x complex plane regardless of the interval on which the eigenvalue problem (1) is posed. According to the general theory of ordinary differential equations, the solution $y(x)$ of (1) can have the singularities only in the points where $U(x)$ is singular. Since $f(x)$ in (1) is an entire function, imposed boundary conditions imply that $y(x)$ is an entire function of x.

In the present paper we consider the potentials belonging to the following class. Let

$$M(r) = \max_{|x|=r} |y(x)|.$$

We consider in (1) the potentials for which $M(r) < \exp(r^{\mu})$ for some finite positive number μ and for all sufficiently large r. In other words, a finite positive number μ exists, such that the maximum of an absolute value of $y(x)$ for large values of $|x|$ grows no faster than $\exp(r^{\mu})$. For example, if in (2) the function $f(x)$ is a polynomial, all the solutions of equation (1) possess this property.

We are going to prove the following statement.

Let the eigenvalue problem (1) have infinitely many eigenvalues belonging to the discrete spectrum with the eigenfunctions possessing the following property. An eigenfunction, considered as a function of a complex variable x, has only a finite number of zeros in a complex plane of a variable x. Then, for the function $f(x)$ from (2) we have two and only two possibilities: $f(x) = ax^3 + bx^2$ or $f(x) = c =$ constant.

The proof is elementary.

Entire functions satisfying inequality $M(r) < \exp r^{\mu}$ for some finite μ and all sufficiently large r are known in the literature as entire functions of a finite order. According to the general theory of entire functions (Markushevitch 1968), any entire function of a finite order $y(x)$ having only a finite number of zeros in the complex x-plane must have the following form,

$$y(x) = P_n(x) \exp Q_m(x) \tag{3}$$

where $P_n(x)$ and $Q_m(x)$ are the polynomials of degree n and m, respectively. We note now that if (1) has a solution (3) then, for sufficiently large $|x|$, the asymptotic form of $y(x)$ is given by the semiclassical WBKJ formula (Landau and Lifshitz 1975). Indeed, $y(x) = P_m \exp Q_n = \exp G$, where $G = Q_n + \ln P_m$. It is easy to see that the conditions of the applicability of the WBKJ approximation ($G'' \ll G'^2$) are always fulfilled for sufficiently large $|x|$. Thus, the asymptotic form of $y(x)$ is given by

$$y(x) \sim (U(x) - E)^{-\frac{1}{4}} \exp\left\{ -\int_0^x \sqrt{2(U(t) - E)}\, \mathrm{d}t \right\} \qquad |x| \to \infty. \tag{4}$$

We emphasize that the asymptotic form (4) is valid everywhere in the complex x-plane for sufficiently large $|x|$. Let us consider first the case when $Q_m(x)$ in (3) is a polynomial of a first degree. It is easy to see that the expression in braces in (4) can give a polynomial of a first degree for all x only if $|U(x)| < C$ for some constant C for all sufficiently large $|x|$. (If there was some ray in a complex x-plane along which $|U(x)| \to \infty$, then the expression in braces in (4) would not give asymptotically the polynomial of a first degree.)

Using this upper bound for $U(x)$ we obtain the bound for $f(x) : |f(x)| < C|x|$ for sufficiently large $|x|$. Since $f(x)$ is an entire function it implies that $f(x) = c + c_1 x$ (according to the Liouville theorem). With the term $c_1 x$ giving only an additive change of an energy scale, we can consider only the case $f(x) = C =$ constant corresponding to the Coulomb potential. As is well known, in this case (if $C < 0$) wavefunctions of the discrete spectrum of an eigenvalue problem (1) have the form (3) and thus have a finite number of zeros in the x complex plane.

Consider now the case of $Q_m(x)$ in (3) with $m > 1$. It can be shown that the expression in braces in (4) can reproduce asymptotically the polynomial of degree $m > 1$ only if the potential $U(x)$ satisfies the following condition everywhere in the complex x-plane: $|U(x)| \to \infty$ when $|x| \to \infty$. Expanding the expression in braces in equation (4) in the powers of $E/U(x)$ one obtains

$$y(x) \sim U(x)^{-\frac{1}{4}} \exp\left\{ \int^x \sqrt{2U(t)} \left(1 - \frac{E}{2U(t)} - \frac{E^2}{8U(t)^2} + \cdots \right) \mathrm{d}t \right\}. \tag{5}$$

From this formula, one can easily see that to ensure the leading x^m behaviour of the expression in braces in (5), the potential $U(x)$ must behave for large $|x|$ as x^{2m-2}. According to the Liouville theorem it implies that $f(x)$ is a polynomial of degree $2m - 1$. If we are interested in the asymptotic form of $y(x)$, we can keep only the two first terms of the expansion in the powers of $E/U(x)$ in the exponential in (5). The higher order terms in $E/U(x)$ give a contribution behaving as $1/U(x)^{\frac{3}{2}} \sim 1/x^{3(m-1)}$. After integration they would be negligible for large $|x|$ corrections. We obtain, thus,

$$y(x) \sim U(x)^{-\frac{1}{4}} \exp\left\{ \int^x \left(\sqrt{2U(t)} - \frac{E}{\sqrt{2U(t)}} \right) \mathrm{d}t \right\}. \tag{6}$$

We note now that, if $m > 2$, we can also omit the second term in the exponential in (6) for large values of $|x|$. Indeed, $E/\sqrt{U(t)} \sim 1/x^{m-1}$ and for $m > 2$ after the integration it gives the correction $\sim 1/x^{m-2}$, negligible for $|x| \to \infty$. For $m > 2$ we therefore have the following asymptotic formula for $y(x)$:

$$y(x) \sim U(x)^{-\frac{1}{4}} \exp\left\{ \int^x \sqrt{2U(t)}\, \mathrm{d}t \right\}. \tag{7}$$

This asymptotic is to be compared with that of the exact solution (3):

$$y(x) \sim x^n \exp Q_m(x). \tag{8}$$

As we have established above, $U(x)$ behaves as x^{2m-2} for large $|x|$. In order to reproduce the factor x^n in (8), the square root $\sqrt{2U(x)}$ in formula (7) should have the following large-x asymptotic expansion,

$$\sqrt{2U(x)} \sim C_{m-1}x^{m-1} + C_{m-2}x^{m-2} + \cdots + \frac{C_{-1}}{x} \tag{9}$$

where the coefficient C_1 satisfies $C_{-1} = n - (1-m)/2$. (The terms decaying faster than $1/x$ can be omitted since they do not contribute to the asymptotic of a pre-exponential factor.) Thus, in the case considered (the degree m is greater than 2), the eigenvalue problem (1) has a solution of type (3) only if the potential $U(x)$ is 'state-dependent', that is its parameters depend on the number of zeros of some particular solution. Therefore, in this case, (1) cannot have infinitely many solutions of type (3).

The last case to consider is that Q_m in formula (3) is a second-order polynomial. In this case, as we have shown above, $U(x) \sim x^2$ for large $|x|$, the function $f(x)$ in (1) is a third-order polynomial, and the potential $U(x)$ in (1) has the following form:

$$U(x) = \frac{l(l+1)}{2x^2} + \frac{C_3x^3 + C_2x^2 + C_1x + C_0}{x}. \tag{10}$$

It is known that the eigenvalue problem (1) with potential (10) has an infinite number of solutions of type (3) in the following cases:

(a) $x \in (-\infty, \infty)$, $l = 0$, $C_0 = 0$, the one-dimensional harmonic oscillator;

(b) $x \in [0, \infty)$, $C_0 = 0$, $C_2 = 0$, the radial equation for the three-dimensional harmonic oscillator. Our assertion is thus proved.

We are now able to show that the harmonic oscillator and the Coulomb potentials are the only potentials, belonging to a class of potentials introduced in the present paper, for which the semiclassical and the exact quantum mechanical quantization give identical results. The coincidence of the semiclassical and the exact results for the harmonic oscillator and the Coulomb problem can be explained as follows (Bertocchi *et al* 1965).

Let us write the well known formula for the number of zeros of an analytic function $y(x)$,

$$n = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{y'(x)}{y(x)}\, \mathrm{d}x \tag{11}$$

where γ is a circle $|x| = R$ and n is the number of zeros of $y(x)$ situated inside γ. The bound state wavefunctions of the harmonic oscillator and the Coulomb problem have only a finite number of zeros in a complex plane. Moreover, all zeros of the bound state wavefunctions are on the real axis.

Let us choose the contour γ in (11) so that all zeros of $y(x)$ are situated inside γ. Then one can deform the circle γ, increasing arbitrarily its radius R. For large values of R one can substitute into the integral the WBKJ asymptotic for $y(x)$. One can see that this substitution introduces an error becoming negligible for $R \to \infty$. Since R can be chosen arbitrarily large, one obtains in this way an exact estimation for the integral (11). Using this result one obtains from (11) the exact discrete spectrum for the one- and three-dimensional harmonic oscillators and the Coulomb problem.

If in (11) the bound-state wavefunction $y(x)$ has zeros in the complex plane the situation is different. The contour γ in (11) has to be chosen in such a way that its interior contains only the real zeros of $y(x)$. This means that the distance from the real axis to the contour γ is finite and is equal to the distance from the real axis to the nearest complex zero of $y(x)$. Since in the region of finite x the WBKJ solution of equation (1) only gives an approximation to the exact solution, formula (11) only gives an approximation to the exact energy spectrum, the quality of approximation being better the larger the distance from the real axis to the nearest complex zero of $y(x)$ (Bertocchi *et al* 1965).

Since, as we saw above, the bound-state wavefunctions for the systems other than the harmonic oscillator and the Coulomb potential always have an infinite number of zeros, $y(x)$ has an infinite number of complex zeros. The semiclassical quantization of these systems therefore gives only an approximation to the exact spectrum.

We summarize our findings as follows. The Coulomb potential, one- and three-dimensional harmonic oscillator potentials are the only potentials, belonging to the class of potentials defined in the present paper, for which the eigenvalue problem (1) has infinitely many discrete spectrum eigenvalues with the eigenfunctions having only a finite number of zeros in a complex plane. It is due to this fact these two potentials are the only potentials belonging to the class defined in the present paper, for which the semiclassical and exact quantum mechanical treatment give the coinciding results.

Acknowledgment

Author is thankful to the referee for the helpful suggestions and for making some references available.

References

Bertocchi L, Fubini S and Furlan G 1965 *Nuovo Cimento* **35** 599
Landau L D and Lifshitz E M 1971 *Mechanics* (Moscow: Mir)
Landau L D and Lifshitz E M 1975 *Quantum Mechanics* (Oxford: Pergamon)
Markushevitch A I 1968 *Theory of Analytic Functions* (Moscow: Mir)